Capital Assets Pricing Model

Wojciech Otto, Assistant Profesor,
Department of Economics, University of Warsaw

1. Portfolio Analysis—Portfolio of Shares with short sale allowed

Let:

$$R_i = \frac{P_{i,1} + D_{i,1} - P_{i,0}}{P_{i,0}}$$

be a rate of return on investment in shares of i-th company for a period of time of unit length, where:

- $P_{i,0}$ and $P_{i,1}$ are prices of one share of i-th company, at the beginning and at the end of the period, respectively (at time $t = 0$ and $t = 1$),
- $D_{i,1}$ is a value of dividends paid during the period (0, 1], accumulated to the time moment $t = 1$.

Let R denote the column vector of rates of return R_i, $i = 1, ..., n$ on investment in shares of (n) various companies.

We treat rates of return as random variables with joint probability distribution, with (vector of) expected values:
$\mu = E(R)$ containing elements $\mu_i = E(R)$,
and a positively defined (hence nonsingular) covariance matrix:
$A = E\{(R-\mu)\cdot(R-\mu)'\}$ containing elements $a_{i,j} = COV(R_i, R_j)$.

Assume now that we invest in shares of the i-th company the amount x_i at the beginning of the period. By the portfolio we mean n-element (column) vector containing elements x_i. Rate of return on investment in the portfolio x equals:

$$R_x = \frac{\sum_{i=1}^{n} x_i \cdot (1+R_i) - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i} = \frac{\sum_{i=1}^{n} x_i \cdot R_i}{\sum_{i=1}^{n} x_i} = \frac{x'R}{x'l}$$

where l denotes n-element (column) vector of ones, and x' denotes the vector x transposed. We will also assume that the amount invested equals one (dollar?):

$$x'l = 1$$

thus the rate of return on portfolio x just equals:

$$R_x = x'R$$

As the rate of return is a linear combination of individual company rates of return, we obtain easily:

$$E(R_x) = x'\mu$$

and:

$$VAR(R_x) = x'Ax$$

We will not assume, that all elements of the vector x are non-negative. Possibility to issue the liability on shares of a given company (to invest the negative amount in its shares) is called "short sale", and in general many stock exchanges allow for such transactions. Allowing for short sale radically simplifies the portfolio analysis—otherwise being a fairly complex (mathematically) problem. The Capital Assets Pricing Model could be derived despite this simplification, so for the purpose of this paper any unnecessary complications are not required.

1.1. Minimum risk portfolio.

Problem 1. Find the portfolio x^* having the minimal variance of the rate of return.

Solution:

Positively defined quadratic form (under the restriction $x'l = 1$) has a unique global minimum which could be found by equating to zero first-order partial derivatives of the function:

$$L(x, \lambda) = x'Ax + \lambda(x'l - 1)$$

where λ is a Lagrange multiplier.

As a result we obtain the system of $(n + 1)$ equations of the form:

$$\begin{cases} 2Ax + \lambda l = 0 \\ x'l = 1 \end{cases}$$

Multiplying both sides of first n equations by the scalar 0.5 and by the inverse of A we obtain:

$$x = \frac{1}{2}\lambda \cdot A^{-1}l$$

Multiplying in turn both sides of first n equations by x' (from the left) we obtain:

$$2x'Ax + \lambda x'l = 0$$

which directly leads to the result:

$$\lambda = -2x'Ax$$

Substituting x by the previous result we obtain:

$$\lambda = -2 \cdot \left(-\frac{1}{2}\lambda \cdot A^{-1}l\right)' \cdot A \cdot \left(-\frac{1}{2}\lambda \cdot A^{-1}l\right) = -\frac{1}{2}\lambda^2 \cdot l'A^{-1}l$$

Which leads to the result:

$$\lambda = -\frac{2}{l'A^{-1}l}$$

(*there is no risk of dividing by zero because of the assumption of A being positively defined*), which allows for the final result, i.e. the minimum-variance portfolio:

$$x_* = \frac{A^{-1}l}{l'A^{-1}l}$$

for which the expectation and variance can be easily calculated as:

$$\begin{cases} \mu_* = x_*'\mu = \dfrac{l'A^{-1}\mu}{l'A^{-1}l} \\ \sigma_*^2 = VAR\left(R_{x_*}\right) = \dfrac{\left(A^{-1}l\right)' AA^{-1}l}{\left(l'A^{-1}l\right)^2} = \dfrac{1}{l'A^{-1}l} \end{cases}$$

which finalises the solution of the problem 1.

Exercise 1:

Assume that the matrix A is of the form:

$$A = \sigma^2\left((1-\rho)\cdot I + \rho \cdot ll'\right)$$

where I denotes the unit matrix $n \times n$. The assumption simply means that variance of the rate of return for each individual company is σ^2, and linear correlation coefficient for any pair of companies equals ρ.

Find the minimum variance portfolio x^*, its expectation and variance.

Remark: for mathematical correctness of these assumptions one should presume that

$$\rho \in \left(-\frac{1}{n-1}, 1\right)$$

From the economic point of view it is sensible to restrict our considerations to the interval $\rho \in [0, 1)$.

Solution: direct application of well known rules of matrix inversion leads to the following result:

$$A^{-1} = \frac{1}{\sigma^2 \cdot (1-\rho)\cdot\left(1+(n-1)\cdot\rho\right)} \cdot \left[\left(1+(n-1)\cdot\rho\right)\cdot I - \rho \cdot ll'\right]$$

- hence all elements of the vector x_* equal $\frac{1}{n}$,

- expected rate of return is just the simple average of individual rates:

$$\mu_* = \frac{1}{n} \cdot l'\mu,$$

- and the minimal variance amounts to:

$$\sigma_*^2 = \frac{1}{l'A^{-1}l} = \sigma^2 \cdot \left(\frac{1}{n} + \frac{n-1}{n} \cdot \rho\right)$$

In the formula for the variance the first term represents the diversifiable part of risk, while along the increasing number of companies n this component tends to vanish. The second component represents the non-diversifiable part of risk, so roughly speaking it corresponds to the general situation of the market, reflected in simultaneously parallel deviations of all individual rates.

1.2. Minimum variance portfolio among portfolios with a given expectation

Let us also assume now that elements of the vector of expectations μ are (at least some of them) mutually different. Of course it means that the number of companies n is at least 2. Then the system of equations:

$$\begin{cases} x'l = 1 \\ x'\mu = y \end{cases}$$

has for an arbitrary real number y a solution (when $n > 2$, then the number of solutions is infinite). Thus under this additional assumption the following problem is well posed.

Problem 2. Find portfolio $x(y)$ having the minimum variance among all portfolios with expected rate of return y, where y is a predetermined arbitrary real number.

As a prelude of solution we will prove at first a lemma:

Lemma 1.

Under previously stated assumptions the following inequality holds:

$$l'A^{-1}l \cdot \mu'A^{-1}\mu > \left(l'A^{-1}\mu\right)^2$$

Proof of the Lemma 1:

Since the matrix A is positively defined, then also its inverse A^{-1} is positively defined. Hence the following statements are true:

a) $l'A^{-1}l > 0$ (*which, as something obvious, has been already utilized in the solution of the problem 1*);

moreover, as elements of the vector μ are (at least some of them) different numbers, they cannot be at the same time equal to zero, so:

b) $\mu'A^{-1}\mu > 0$

c) thus there exists such a number $a \neq 0$, that $l'A^{-1}l = a^2 \cdot \mu'A^{-1}\mu$,

d) and hence as well the inequality:

$(l - a\mu)' A^{-1}(l - a\mu) > 0$
as the other one:
$(l + a\mu)' A^{-1}(l + a\mu) > 0$
are true (because neither the vector ($l - a\mu$) nor the vector ($l + a\mu$) is equal to the vector of zeros).

As both quadratic forms quoted in the point d) are positive, we can conclude that:

$$l'A^{-1}l + a^2 \cdot \mu'A^{-1}\mu > |2a \cdot l'A^{-1}\mu|$$

which, taking into account the point c), leads to the conclusion:

$$l'A^{-1}l > |a \cdot l'A^{-1}\mu| \text{ and}$$
$$a^2 \cdot \mu'A^{-1}\mu > |a \cdot l'A^{-1}\mu|$$

Thus the product of left-hand-sides of the two last inequalities is greater than the product of their right-hand-sides:

$$a^2 \cdot \mu'A^{-1}\mu \cdot l'A^{-1}l > a^2 \cdot \left(l'A^{-1}\mu\right)^2$$

which finalises the proof of the Lemma (remind that $a \neq 0$).

Solution of the problem 2.

Now we should equate to zero first order partial derivatives of the function:

$$L(x, \lambda_1, \lambda_2) = x'Ax + \lambda_1(x'l - 1) + \lambda_2(x'\mu - y)$$

The resulting system of (n + 2) equations is of the form:

$$\begin{cases} 2Ax + \lambda_1 l + \lambda_2\mu = 0 \\ x'l = 1 \\ x'\mu = y \end{cases}$$

Similarly as in the problem 1, we now multiply both sides of first n equations by 0.5 and by the inverse of A (from the left):

$$(*)\ x = \frac{1}{2} A^{-1}(\lambda_1 l + \lambda_2\mu)$$

Next multiplying

$$1 = -\frac{1}{2}\left(\lambda_1 l'A^{-1}l + \lambda_2 l'A^{-1}\mu\right)$$

whereas multiplying both sides of the system (*) by μ' (from the left) we obtain:

$$y = -\frac{1}{2}\left(\lambda_1\mu'A^{-1}l + \lambda_2\mu'A^{-1}\mu\right)$$

The last two equations form a system with two unknowns, which could be presented in the form:

$$\begin{cases} \lambda_1 = 2 \cdot \dfrac{y \cdot l'A^{-1}\mu - \mu'A^{-1}\mu}{l'A^{-1}l \cdot \mu'A^{-1}\mu - \left(l'A^{-1}\mu\right)^2} \\ \lambda_2 = 2 \cdot \dfrac{l'A^{-1}\mu - y \cdot l'A^{-1}l}{l'A^{-1}l \cdot \mu'A^{-1}\mu - \left(l'A^{-1}\mu\right)^2} \end{cases}$$

where the denominator of right-hand-side expressions, by virtue of the Lemma 1, differs from zero.

Replacing λ_1 and λ_2 in the system of equations (*) by obtained expressions, we finally obtain the solution:

$$x(y) = \frac{\left(\mu'A^{-1}\mu - y \cdot l'A^{-1}\mu\right)}{l'A^{-1}l \cdot \mu'A^{-1}\mu - \left(l'A\mu\right)^2} \cdot A^{-1}l + \frac{\left(y \cdot l'A^{-1}l - l'A^{-1}\mu\right)}{l'A^{-1}l \cdot \mu'A^{-1}\mu - \left(l'A^{-1}\mu\right)^2} \cdot A^{-1}\mu$$

which represents points belonging to the straight line (one-dimensional hyperspace of the n-dimensional space). It is easy to perceive that each point is a linear combination of the vector $A^{-1}l$ and the vector $A^{-1}\mu$, and that coefficients of this linear combination are linear functions of the variable y. Two different points stretching this line could be easily obtained by substituting the variable y by two different fixed numbers (for example zero and one). Of course, by substitution $y = \mu_*$ we obtain $x(y) = x_*$.

Variance of rate of return on portfolio $x(y)$ is given by inserting the result in the general formula:

$$\left[x(y)\right]' \cdot A \cdot x(y)$$

which after tedious calculations leads to the result:

$$\left[x(y)\right]' \cdot A \cdot x(y) = \frac{l'A^{-1}l \cdot y^2 - 2l'A^{-1}\mu \cdot y + \mu'A^{-1}\mu}{l'A^{-1}l \cdot \mu'A^{-1}\mu - \left(l'A^{-1}\mu\right)^2}$$

which is a quadratic function of the real variable y.

1.3. Admissible portfolios, effective portfolios

Let us assume now that the analysis is performed on request of an investor whose preferences could be reflected by two simple rules:

- out of two portfolios of equal expectations the one of smaller variance is preferred,
- out of two portfolios of equal variances the one of greater expectation is preferred.

A d m i s s i b l e p o r t f o l i o is any point (element) of the set:

$$\{x:\ x \in R^n,\ l'x = 1\}$$

E f f e c t i v e p o r t f o l i o is such an admissible portfolio for which there is no better (in the sense given by preferences described above) admissible portfolio.

Let us introduce a two-dimensional function which attributes to each portfolio its variance and expectation. More precisely, the function attributes to each point of the set $\{x:\ x \in R^n,\ l'x = 1\}$ the point on the plane with coordinates $(x'Ax,\ x'\mu)$. Such a function is often called *mapping*.

We know about this *mapping* that:

- for each real number y set of all portfolios x such that their expectation equals y (so satisfying the condition $x'\mu = y$) is *mapped* to the set of points of the form $\left(\sigma^2(x),\ y\right)$, where we know that the first coordinate attains its minimum for the portfolio $x(y)$ (the formula has been obtained when solving the Problem 2), and that this minimum amounts to:

$$\left[x(y)\right]' \cdot A \cdot x(y) = \frac{l'A^{-1}l \cdot y^2 - 2 \cdot l'A^{-1}\mu \cdot y + \mu'A^{-1}\mu}{l'A^{-1}l \cdot \mu'A^{-1}\mu - \left(l'A^{-1}\mu\right)^2}$$

  so is a quadratic function of the real variable y.
- From the above it results that the image of the whole set of admissible portfolios under its mapping into the plane $\left(x'A^{-1}x,\ x'\mu\right)$ is the subset of the plane cut out by parabola, which may be defined as follows:

$$\left\{(z,\ y):\ (z,\ y) \in R^2,\ z \geq \left[x(y)\right]' \cdot A \cdot x(y)\right\} \text{ provided } n > 2$$

- In the special case, when $n = 2$, then the image of mapping contains only points lying on the parabola. When $n = 3$, then each point of the parabola corresponds to exactly one portfolio, whereas each point lying to the right of the parabola line corresponds to two different portfolios. When $n > 3$, then still points lying on the parabola line correspond to uniquely defined portfolios (given in the solution of the Problem 2), whereas the counter-image of any point lying to the right of the parabola line is an infinite set of points in the R^n space.

Traditionally results of the analysis are presented on the graph, where expectations are on the vertical axis, whereas on the horizontal axis standard deviations, instead of variances, are laid off. Then the image of mapping is a set:

$$\left\{(z,\ y):\ (z,\ y) \in R^2,\ z \geq \sqrt{\left[x(y)\right]' \cdot A \cdot x(y)}\right\}$$

which has a shape of hyperbola (together with points to the right of the hyperbola line). More precisely, it means the right arm of the hyperbola (the whole image is contained in the half-plane right to the vertical axis). Coordinates of the apex of the right arm of the hyperbola are (σ_*, μ_*), given by the solution of the Problem 1. To find out asymptotes we need to notice that:

- they have to cross one another at the point $(0, \mu_*)$
- their slope coefficients are (in terms of absolute values) equal to the limit:

$$\lim_{y \to \infty} \frac{\partial}{\partial y} \left(\sqrt{[x(y)]' \cdot A \cdot x(y)} \right)$$

Simple (although tedious) calculations lead to the conclusion that both asymptotes could be expressed by one formula:

$$\left\{ (z, y): \ (z, y) \in R^2, \ \ y = \mu_* \pm z \cdot \sqrt{\mu' A^{-1} \mu - \frac{(l' A^{-1} \mu)^2}{l' A^{-1} l}} \right\}$$

Of course only such portfolio which minimizes variance (given the expectation) may belong to the set of efficient portfolios. However, not all solutions of the Problem 2 are efficient. Out of the set of all solutions we have to eliminate those portfolios for which (given the variance) there are other portfolios with higher expectations. On that plane, on which expectations and standard deviations are laid off, the image of the set of all solutions of the Problem 2 lies on the right-hand-side arm of the hyperbola which is symmetric in respect of the horizontal line crossing the arm at the apex $(0, \mu_*)$. Thus the image of the set of efficient portfolios contains only that part of the right arm of the hyperbola which lies above the level of the apex (so called e f f i c i e n t f r o n t i e r). The set can be described as follows:

$$\left\{ (z, y): \ (z, y) \in R^2, \ \ z = \sqrt{[x(y)]' \cdot A \cdot x(y)}, \ \ y \geq \mu_* \right\}$$

and its counter-image (in the portfolio space) is a set of such $x \in R^n$, that:

$$x = \frac{(\mu' A^{-1} \mu - y \cdot l' A^{-1} \mu)}{l' A^{-1} l \cdot \mu' A^{-1} \mu - (l' A^{-1} \mu)^2} \cdot A^{-1} l + \frac{(y \cdot l' A^{-1} l - l' A^{-1} \mu)}{l' A^{-1} l \cdot \mu' A^{-1} \mu - (l' A^{-1} \mu)^2} \cdot A^{-1} \mu$$

where $y \in (\mu_*, \infty)$.

Of course this set has a form of the ray (included in the space R^n).

Figure 1. contains a graph of the right arm of the hyperbola line, pointing out the efficient frontier, and asymptotes of the hyperbola. That part of the hyperbola line, which corresponds to portfolios with all elements of the vector x being non-negative is also shown. It is easy to find that part of the hyperbola:

$$\left\{ (z, y): \ (z, y) \in R^2, \ \ z = \sqrt{[x(y)]' \cdot A \cdot x(y)}, \ \ x_i(y) \geq 0, \, i = 1, 2, \ldots, n \right\}$$

as all elements of the vector x are linear functions of the same variable y (it could of course happen that for a given matrix A and vector μ this set is an empty set). The graph contains also other elements, which are explained in the next section. The graph represents the case when $n = 3$, expected rates of return are $\mu' = (0.20\ 0.15\ 0.10)$, variances of rates are $(a_{11}\ a_{22}\ a_{23}) = (0.20\ 0.15\ 0.10)$, and for each pair of rates the linear correlation coefficient equals 0.4.

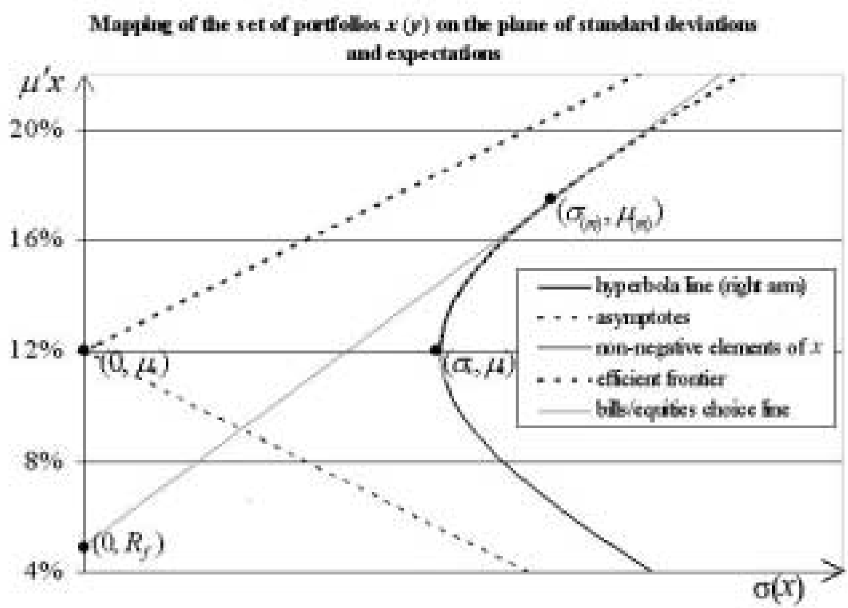

Figure 1.

Image of the set of portfolios being solutions of the problem 2 on its mapping into the plane of standard deviations and expected values of the rate of return.

Exercise 2.

Let us assume, as in the exercise 1, that A is of the form:

$$A = \sigma^2\left((1-\rho)\cdot I + \rho l l'\right), \text{ where } \rho \in [0, 1).$$

Let us recollect that under this assumption the apex of the hyperbola has coordinates:

$$\left(\sqrt{\sigma^2(1-\rho)\cdot\left(\frac{1}{n}+\frac{n-1}{n}\cdot\rho\right)},\ \bar{\mu}\right), \text{ where } \bar{\mu} = \frac{l'\mu}{n}$$

Asymptotes in turn could be expressed by formula:

$$\left\{(z,\ y):\ (z,\ y)\in R^2,\ \ y = \bar{\mu} \pm z\cdot\sqrt{\frac{1}{\sigma^2(1-\rho)}\cdot\left(\mu'\mu - n\cdot\bar{\mu}^2\right)}\right\}$$

which could be re-expressed in the equivalent form:

$$\left\{(z,\ y):\ (z,\ y)\in R^2,\ \ y=\bar{\mu}\pm z\cdot\sqrt{\frac{1}{\sigma^2(1-\rho)}\cdot\sum_{i=1}^{n}(\mu_i-\bar{\mu})^2}\right\}$$

It is recommended to the reader to inspect (on the basis of the above formula), the influence of parameters σ^2, ρ, n and

$$\frac{1}{n}\sum_{i=1}^{n}(\mu_i-\bar{\mu})^2$$

on the shape of the hyperbola, given by the gradient of asymptotes.

2. Portfolio analysis—investing in shares and bonds when short sale is allowed

Let us assume that the real alternative to investing in equities is the possibility to invest in securities which are risk free (rate of return is R_f with probability one).

For example, if our analysis focuses on one-week rate of return, the availability of treasury bills (or some other securities free of default risk) with one-week time to maturity is required. Our decision problem can be formulated now as the problem of choice of a pair:

$$(\delta,\ x):\ \ \delta\in R,\ \ x\in R^n,\ \ l'x=1$$

where δx_i is the amount invested in i-th company, whereas $(1-\delta)$ is the amount invested in treasury bills.

Expected rate of return for such portfolio amounts to:

$$E\left(R_{\delta,x}\right)=\delta\cdot x'\mu+(1-\delta)\cdot R_f$$

whereas the standard deviation to:

$$\sqrt{VAR\left(R_{\delta,x}\right)}=|\delta|\cdot\sqrt{x'Ax}$$

Let us fix now the internal structure of sub-portfolio of shares x such that $x'\mu > R_f$ (such x exists for arbitrary predetermined R_f, provided that elements of the vector m are not all identical—what we have already assumed).

Then for $\delta \geq 0$ both the expectation and the variance of the rate of return on portfolio (δ, x) are increasing functions of δ, and each such portfolio could possibly be an efficient one. Any portfolio with the coefficient $\delta < 0$ cannot be efficient (has smaller expectation and greater variance than the portfolio with $\delta = 0$).

Therefore portfolios potentially efficient are represented on the plane by points belonging to the ray:

$$\left(0, R_f\right) + \delta \cdot \left(\sqrt{x'Ax},\ \left(x'\mu - R_f\right)\right), \text{ where } \delta \geq 0.$$

The position of the apex of the ray $(0, R_f)$ is always the same. However, the choice of the structure of the sub-portfolio of shares x influences the slope of the ray. Obviously, the greater slope coefficient the better, as the slope coefficient represents how much expectation increase in exchange of unit increase of the standard deviation. Such sub-portfolio structure x, for which the slope will be the highest, is the solution to the following problem:

Problem 3.

Find x for which the function:

$$g(x) = \frac{x'\mu - R_f}{\sqrt{x'Ax}} \text{ attains its maximum,}$$

under the restriction $x'l = 1$.

Solution.

The Lagrangian takes now the form:

$$L(x, \lambda) = \frac{x'\mu - R_f}{\sqrt{x'Ax}} + \lambda \cdot \left(x'l - 1\right)$$

Equating its first order partial derivatives to zero we obtain the system of equations:

$$\begin{cases} \dfrac{x'Ax \cdot \mu - \left(x'\mu - R_f\right) \cdot Ax}{\left(x'Ax\right)^{1.5}} = -\lambda \cdot l \\ x'l = 1 \end{cases}$$

Multiplying now both sides of the subsystem of first n equations (from the left) by x' we obtain:

$$\lambda = \frac{-R_f}{\sqrt{x'Ax}}$$

which, introduced back to the first n equations gives the result:

(**) $$x'Ax \cdot \left(\mu - R_f \cdot l\right) = \left(x'\mu - R_f\right) \cdot Ax$$

Let us also assume now, that $l'A^{-1}(\mu - R_f \cdot l) \neq 0$.

Multiplying now both sides of (**) from the left by $l'A^{-1}$, and dividing them by the expression $l'A^{-1}(\mu - R_f \cdot l)$, we obtain:

$$x'Ax = \frac{\left(x'\mu - R_f\right)}{l'A^{-1}\left(\mu - R_f \cdot l\right)}$$

which again introduced to the system of equations (**) leads to the result:

$$Ax = \frac{\left(\mu - R_f \cdot l\right)}{l'A^{-1}\left(\mu - R_f \cdot l\right)}$$

which directly leads to the solution x_0:

$$x_0 = \frac{A^{-1}\left(\mu - R_f \cdot l\right)}{l'A^{-1}\left(\mu - R_f \cdot l\right)}$$

So far we have no guaranty that the obtained solution really maximizes the slope coefficient (obtained point x_0 satisfies only the first-order conditions). In order to have a closer look on the solution, we should find its image on the plane of expectations and standard deviations. The expectation amounts to:

$$x_0'\mu = \frac{\mu'A^{-1}\left(\mu - R_f \cdot l\right)}{l'A^{-1}\left(\mu - R_f \cdot l\right)}$$

which could be also re-expressed in the form:

$$x_0'\mu = \frac{\mu'A^{-1}\mu - R_f \cdot \mu'A^{-1}l}{l'A^{-1}\mu - R_f \cdot l'A^{-1}l} =$$

$$= \frac{\dfrac{\mu'A^{-1}\mu \cdot l'A^{-1}l - \left(\mu'A^{-1}l\right)^2 + \left(\mu'A^{-1}l\right)^2}{\left(l'A^{-1}l\right)^2} - R_f \dfrac{\mu'A^{-1}l}{l'A^{-1}l}}{\dfrac{l'A^{-1}\mu}{l'A^{-1}l} - R_f} =$$

$$= \frac{\dfrac{\mu'A^{-1}\mu \cdot l'A^{-1}l - \left(\mu'A^{-1}l\right)^2}{\left(l'A^{-1}l\right)^2} + \mu_*^2 - R_f \cdot \mu_*}{\mu_* - R_f}$$

which finally could be expressed as the sum of two components:

$$x_0'\mu = \mu_* + \frac{\mu'A^{-1}\mu \cdot l'A^{-1}l - \left(\mu'A^{-1}l\right)^2}{\left(l'A^{-1}l\right)^2 \cdot \left(\mu_* - R_f\right)}$$

By the virtue of the Lemma 1 (section 1.2) the numerator of the second component is positive. Thus we could conclude that:

$$x_0'\mu > \mu_* \Leftrightarrow R_f < \mu_*$$

and:

$$x_0'\mu < \mu_* \Leftrightarrow R_f > \mu_*$$

Now we are ready to understand better the essence and geometry of the problem, which we have just solved.

If $R_f < \mu_*$, then the point $(0, R_f)$ lies below the point $(0, \mu_*)$, at which asymptotes of the parabola cross. So it is possible to find a ray with the apex at $(0, R_f)$, having a positive slope, and being tangent to this part of the hyperbola, which we have pointed out (when solving Problem 2) as being the set representing efficient sub-portfolios of shares. In this case the obtained solution of the problem 3 is in line with our general task.

On the other hand, if $R_f > \mu_*$, then the point $(0, R_f)$ lies above the point $(0, \mu_*)$, so the ray with apex at $(0, R_f)$, and tangent to the right arm of the hyperbola could be tangent only to the lower part of the right arm—but points lying on the lower part correspond to inefficient portfolios.

Still, if $R_f = \mu_*$, then there are no rays with apex at $(0, R_f)$, tangent to the hyperbola. This is because any ray with a slope greater or equal to the slope of the asymptote will have no common points with the hyperbola—whereas any ray with smaller slope will cross the hyperbola line, so it could be replaced by a "better ray" having at least a little bit greater slope. More precisely, if the slope is smaller than the slope of the asymptote, then there exists such positive number ε, that the slope could be expressed as:

$$\sqrt{\mu' A^{-1}\mu - \frac{\left(l' A^{-1}\mu\right)^2}{l' A^{-1} l}} - \varepsilon$$

then the ray will obviously cross the hyperbola line. However, the same could be said about the ray with the slope equal to:

$$\sqrt{\mu' A^{-1}\mu - \frac{\left(l' A^{-1}\mu\right)^2}{l' A^{-1} l}} - \varepsilon \cdot \left(\frac{1}{2}\right)^n$$

where n is an arbitrary positive integer. It is clear that for increasing n crossing points of subsequent rays with the hyperbola will move to the right and upwards, towards portfolios with arbitrary large expectations and standard deviations.

In the case when $R_f = \mu_*$ it is also impossible to point out the ray being tangent to the lower part of the right arm of the hyperbola (arguments are analogous).

Summarizing the solution of the Problem 3, effective sub-portfolio of shares could be pointed out only in the case when $R_f < \mu_*$. Then it is given by the formula:

$$x_0 = \frac{A^{-1}\left(\mu - R_f \cdot l\right)}{l' A^{-1}\left(\mu - R_f \cdot l\right)}$$

and its image on the plane of expectations and standard deviations of rates of return is the point:

$$\left(\sqrt{x_0' A x_0}, x_0'\mu\right)$$

This image could be compared with the image of the minimum-variance portfolio, after re-expression to the form:

$$x_0'\mu = \mu_* + \frac{\mu' A^{-1}\mu \cdot l' A^{-1} l - \left(\mu' A^{-1} l\right)^2}{\left(l' A^{-1} l\right)^2 \cdot \left(\mu_* - R_f\right)}$$

$$\sqrt{x_0' A x_0} = \sigma_* \cdot \sqrt{1 + \frac{\mu' A^{-1}\mu \cdot l' A^{-1} l - \left(\mu' A^{-1} l\right)^2}{\left(l' A^{-1} l\right)^2 \cdot \left(\mu_* - R_f\right)}}$$

3. Capital Assets Pricing Model (CAPM) and the structure of rates of return on investment in shares

Let us now assume that all investors present on the market:

- are driven by preferences described at the beginning of the section 1.3,
- analyse data on the basis of the same time interval (for example all are interested in rates of return on various assets over one month),
- have at disposal the same set of information about the market (matrix A and vector μ corresponding to the same basic time interval),
- undertake investment decisions according to results of the analysis, by zero transaction costs;

and that the market is in equilibrium.

Then each investor will choose the effective portfolio, which is represented on the expectations—standard deviations plane by the point of the ray:

$$\left(0, R_f\right) + \delta \cdot \left(\sqrt{x_0' A x_0}, \left(x_0'\mu - R_f\right)\right)$$

The important conclusion is that portfolios of all investors in the part of shares will have an identical structure given by the vector x_0. Depending on preferences, individual investors will differ only by proportion of sub-portfolio of bills and sub-portfolio of shares.

However, since all investors will invest in shares maintaining the same structure and the market is in equilibrium, then this structure has to be identical to the structure of supply (in terms of values). Denoting by x_m the market portfolio (the supply structure) we can express the equilibrium assumption in the form: $x_m = x_0$.

Taking into account the result (**) (obtained when solving the Problem 3), in equilibrium the following system of equations holds:

$$x_m' A x_m \cdot \left(\mu - R_f \cdot l\right) = \left(x_m'\mu - R_f\right) \cdot A x_m$$

which can be re-expressed as:

$$\mu = R_f \cdot l + \left(\mu_{(m)} - R_f\right) \cdot \frac{A x_m}{\sigma_{(m)}^2}$$

where the simplifying notations for expectation and variance for the market portfolio are used:

$$\mu_{(m)} = x'_m \mu$$
$$\sigma^2_{(m)} = x'_m A x_m$$

The individual equation (let us say, number i) out of the above system of equations can be written as:

$$\mu_i = R_f + \left(\mu_{(m)} - R_f \right) \cdot \frac{A_i x_m}{\sigma^2_{(m)}}$$

where A_i is an i-th row of the matrix A.

However, the row contains covariances of rate of return on investment in shares of company number i with rates of return of subsequent (from 1 to n) companies. Hence we obtain:

$$A_i x_m = \sum_{j=1}^{n} COV\left(R_i, R_j\right) \cdot x_{m,j} = COV\left(R_i, R_{x_m}\right)$$

Finally we get the result which states that in equilibrium the rate of return on investment in shares of the company number i accounts to:

(***) $$\mu_i = R_f + \left(\mu_{(m)} - R_f \right) \cdot \beta_i$$

where

$$\beta_i = \frac{A_i x_m}{\sigma^2_{(m)}}$$

is simply the regression coefficient in the linear regression model:

$$R_i = R_f + \left(R_{x_m} - R_f \right) \cdot \beta_i + \varepsilon_i$$

according to which the total variance of the rate of return $a_{i,i}$ (corresponding element of the matrix A) is decomposed into two parts:

- covariance with the rate of return on market portfolio,
- the lasting component (variance of the individual random deviation ε, uncorrelated with the market portfolio).

Since the first component represents non-diversifiable risk, the rate of return (desired by investors) reflects it by the "risk premium" of amount $(\mu_{(m)} - R_f) \cdot \beta_i$. Since the second component represents the well diversifiable risk (provided n is large), it does not enlarge the expected rate of return.

Expected rates of return imply prices of shares. When for a given company the observed rate of return is higher than expected then the current price is lower than equilibrium price (and vice versa). That is why the equation (***) together with all assumptions needed to derive it is often called the Capital Assets Pricing Model.

4. CAPM and the level of rate of return on investment in shares

One should remark, that the equilibrium assumption releases various complications, which have arisen when solving the Problem 3.

- Firstly, the market portfolio represents supply, so it cannot contain negative elements. Thus we can conclude, that in equilibrium (provided other CAPM assumptions also hold) we will not encounter short sale transactions, even if security market regulations allow for them.
- Secondly, since in equilibrium the optimum structure of sub-portfolio of shares $x_0 = x_m$ is unique, then also the condition $R_f < \mu_*$ must hold.

Unfortunately, the model does not explain how large the difference $(\mu_{(m)} - R_f)$ is. In order to ascertain that, let us inspect once again the system of equations setting the rates of return:

$$\mu - R_f \cdot l = \left(x_m'\mu - R_f\right) \cdot \frac{Ax_m}{\sigma_{(m)}^2}$$

and assume also that (given the supply) there exists a vector $\mu^{(0)}$ which satisfies these equations. Then for an arbitrary real number a the following vector:

$$\mu^{(1)} = a \cdot \mu^{(0)} + R_f \cdot (1 - a) \cdot l$$

also satisfies the same equations, and if $a > 0$, then it is in accordance with CAPM assumptions. However, when replacing the vector $\mu^{(0)}$ by $\mu^{(1)}$, we change also the difference

$$\left(\mu_{(m)} - R_f\right) \text{because } \left(x_m'\mu^{(1)} - R_f\right) = a \cdot \left(x_m'\mu^{(0)} - R_f\right)$$

The fact illustrated above should not be surprising. The value of difference $(\mu_{(m)} - R_f)$ as compared to the standard deviation $\sigma_{(m)}$ sets the substitution rate —shows how large increase of expectation is desired by investors in exchange of unit increase of risk (as measured by standard deviation). It should be reminded that so far we have made no assumption about the substitution rate. All what we have assumed is that given the expectation investors prefer smaller variance, and given variance they prefer greater expectation.

Let us assume now that investors maximise expected utility, and that their utility functions are concave.

Let us consider an investor with utility function $u(\cdot)$ twice differentiable, with derivatives satisfying the following conditions:

$$u'(x) > 0, \quad u''(x) < 0, \quad x \in (-\infty, \infty)$$

having at the starting point savings of amount $K > 0$. Decision problem he (or she) faces, is such a choice of the coefficient d, which maximises the function:

$$g(\delta) = E\left(u\left\{K \cdot \left[1 + R_f + \delta \cdot \left(\xi - R_f\right)\right]\right\}\right)$$

where the random variable ξ, representing the rate of return on investment in the market portfolio, has expectation equal to $\mu_{(m)}$ (which is greater than R_f), and variance $\sigma^2_{(m)}$.

First derivative of the function is of the form:

$$g'(\delta) = E\left(u'\left\{K \cdot \left[1 + R_f + \delta \cdot \left(\xi - R_f\right)\right]\right\} \cdot \left\{K \cdot \left(\xi - R_f\right)\right\}\right)$$

and at the point $\delta = 0$ equals

$$g'(0) = u'\left\{K \cdot \left[1 + R_f\right]\right\} \cdot \left\{K \cdot \left(\mu_{(m)} - R_f\right)\right\}$$

what is obviously a positive quantity.

The second-order derivative of g is of the form:

$$g''(\delta) = E\left(u''\left\{K \cdot \left[1 + R_f + \delta \cdot \left(\xi - R_f\right)\right]\right\} \cdot \left\{K^2 \cdot \left(\xi - R_f\right)^2\right\}\right)$$

and (for arbitrary $\delta > 0$) is negative.

Hence it can be concluded, that either there exists such a number $\delta_0 \in (0, \infty)$, for which the expected utility attains its maximum, or such a number does not exist (for any number δ one can point out a number greater than δ, which leads to the higher expected utility).

Perhaps it could be interpreted as follows:

- The existence of a marginal number of investors (with marginal capital at disposal), whose preferences lead to increasing unboundedly the coefficient δ in their portfolios would probably have no influence on the market because of their limited credibility.
- Substantial number of investors (with substantial amounts of capital) wishing to increase δ as much as possible (borrowing capital at risk free rate R_f to invest it in shares) will influence the market, pushing it towards new equilibrium characterized by lower prices of bills and higher prices of shares—thus reducing the difference $(\mu_{(m)} - R_f)$.

* * *

Readers are generally encouraged to try to manipulate model parameters, to understand how the model will react on exogenous shocks. However, the reader should be aware that probably not much could be derived out from the model, unless particular types of utility functions (as von Neumann–Morgenstern, exponential etc.) are assumed.